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Shape invariance with application to the momentum representation

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Abstract. We formulate the method of shape invariance of supersymmetric potentials in a manner that is suitable for application to spherically symmetric problems in the momentum representation. Three examples are discussed: the isotropic non-relativistic oscillator, a relativistic oscillator, and the Coulomb problem. In these shape invariance is used to determine eigenvalues and normalized momentum-space eigenfunctions for bound states.

1. Introduction

In recent years there has been considerable interest in the use of supersymmetry [1] and shape invariance [2] in quantum mechanics [3-9]. The concept of shape invariance is a sufficient (but not a necessary) condition for the construction of exactly solvable potentials [3, 8], and it has been applied to many of the solvable systems in quantum mechanics [2-6]. Use of shape invariance enables one to determine eigenvalues and eigenfunctions [2-6]: the latter application is essentially a generalization of the standard harmonic oscillator method of raising and lowering operators.

In particular, for the two-body problem with spherically symmetric interparticle potential it has been shown that the three-dimensional non-relativistic isotropic harmonic oscillator and the Coulomb problem are shape invariant [3-5, 9]. In these studies the method of shape invariance is applied in the coordinate representation, and consequently it is the coordinate-space eigenfunctions that are determined.

It is natural to enquire whether shape invariance can also be applied to spherically symmetric potentials in the momentum representation. The purpose of this paper is to show that for certain problems such an application can be made. In section 2 we give a brief review of shape invariance as it is used for spherically symmetric problems in the coordinate representation. In section 3 we formulate the method of shape invariance in a manner that is suitable for application to certain spherically symmetric problems in the momentum representation. Then three examples are discussed: the isotropic non-relativistic oscillator and a relativistic (Dirac) oscillator (section 4), and the Coulomb problem (section 5). For these problems we use the formulation of section 3 to determine energy eigenvalues and normalized momentum-space eigenfunctions for bound states.

2. Shape invariance in the coordinate representation [2–6, 9]

Let $\boldsymbol{r}$ and $\boldsymbol{p}$ denote the position and momentum operators of a particle of mass m. Let

p_r denote the usual radial momentum operator

$$p_r = \frac{1}{2}(\hat{\boldsymbol{r}} \cdot \boldsymbol{p} + \boldsymbol{p} \cdot \hat{\boldsymbol{r}}) = \frac{1}{r}(\boldsymbol{r} \cdot \boldsymbol{p} - \mathrm{i}\hbar) \tag{1}$$

where $r = (\boldsymbol{r} \cdot \boldsymbol{r})^{1/2}$. The operator p_r satisfies the commutation relation (see [9, section 5.2] and [10])

$$[p_r, f(r)] = -\mathrm{i}\hbar \frac{\mathrm{d}f}{\mathrm{d}r}. \tag{2}$$

In the coordinate representation p_r can be expressed as the first-order differential operator

$$p_r = -\frac{\mathrm{i}\hbar}{r}\frac{\mathrm{d}}{\mathrm{d}r} r. \tag{3}$$

Consider the operators

$$A_\pm(\alpha_0) = \pm \frac{\mathrm{i}}{\sqrt{2m}} p_r + W(r, \alpha_0). \tag{4}$$

Here α_0 is a set of parameters and W (the superpotential) is to be determined: it is a real function of r and α_0 (see below). In terms of $A_\pm$ one can construct the supersymmetric partner Hamiltonians

$$H_\pm = A_\pm(\alpha_0)A_\mp(\alpha_0) = \frac{1}{2m} p_r^2 + V_\pm(r, \alpha_0) \tag{5}$$

where

$$V_\pm(r, \alpha_0) = W^2(r, \alpha_0) \pm \frac{\hbar}{\sqrt{2m}} \frac{\mathrm{d}W(r, \alpha_0)}{\mathrm{d}r}. \tag{6}$$

In the coordinate representation $A_\pm$ and $H_\pm$, respectively, are first- and second-order differential operators. The partner potentials $V_\pm$ are said to be shape invariant if [2]

$$V_+(r, \alpha_0) = V_-(r, \alpha_1) + R(\alpha_1) \tag{7}$$

where the remainder R is independent of r and α_1 is some function of α_0: $\alpha_1 = F(\alpha_0)$.

Suppose H_- possesses a discrete spectrum and let $\psi_N(r, \alpha_0)$ denote a normalized coordinate-space eigenfunction with eigenvalue E_N:

$$H_-\psi_N(r, \alpha_0) = E_N\psi_N(r, \alpha_0) \qquad (N = 0, 1, \ldots). \tag{8}$$

For our purposes it is sufficient to suppose that A_+ annihilates ψ_0 (see e.g. [11]):

$$A_+(\alpha_0)\psi_0(r, \alpha_0) = 0. \tag{9}$$

From (5), (8) and (9) it follows that $E_0 = 0$.

If the shape invariance condition (7) is satisfied then the eigenvalues of H_- are given in terms of the remainder R by the simple formula

$$E_N = \sum_{s=1}^{N} R(\alpha_s) \qquad (N = 1, 2, \ldots) \tag{10}$$

where $\alpha_s = F(\alpha_{s-1})$. Also, the operator $A_-(\alpha_0)$ performs the transformation

$$A_-(\alpha_0)\psi_N(r, \alpha_1) = \mathrm{e}^{\mathrm{i}\theta}\sqrt{E_{N+1}}\psi_{N+1}(r, \alpha_0) \tag{11}$$

$(N = 0, 1, \ldots)$, where θ is a real constant. The eigenfunctions of H_- can all be generated from $\psi_0(r, \alpha_N)$:

$$\psi_N(r, \alpha_0) = e^{-iN\theta}\left[\prod_{s=1}^{N}(E_N - E_{s-1})\right]^{-1/2} A_-(\alpha_0)A_-(\alpha_1)\ldots A_-(\alpha_{N-1})\psi_0(r, \alpha_N) \qquad (12)$$

for $N = 1, 2, \ldots$. The inclusion of the phase θ in (12) is necessary in certain problems, such as those involving Dirac spinors [12]. Often the energy-dependent factor in (12) is omitted [2-6]; however, this factor is necessary if we wish to obtain normalized eigenfunctions. Equation (12) is a generalization of the standard operator method for generating the energy eigenfunctions of a one-dimensional harmonic oscillator from its ground state.

The above theory applies also to a spherically symmetric two-body problem if we identify $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$,

$$\boldsymbol{p} = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1 + m_2}$$

and $m = m_1m_2/(m_1 + m_2)$, the reduced mass. For such a problem in an angular momentum basis, H_- is (to within a constant term) the radial Hamiltonian

$$H_l = \frac{1}{2m}p_r^2 + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) \qquad (13)$$

where $V(r)$ is the interparticle potential and l $(=0, 1, \ldots)$ is the orbital angular momentum quantum number. Two important problems to which the method of shape invariance applies are the istropic harmonic oscillator potential

$$V(r) = \tfrac{1}{2}m\omega^2 r^2 \qquad (14)$$

and the attractive Coulomb potential

$$V(r) = -k/r \qquad (15)$$

(k a positive constant). For the oscillator the superpotential is

$$W(r, \alpha_0) = \sqrt{\frac{1}{2}m}\,\omega r - \frac{\hbar(\alpha_0 + 1)}{\sqrt{2m}\,r} \qquad (16)$$

and for the Coulomb problem it is

$$W(r, \alpha_0) = \sqrt{\frac{1}{2}m}\,\frac{k}{\hbar(\alpha_0 + 1)} - \frac{\hbar(\alpha_0 + 1)}{\sqrt{2m}\,r}. \qquad (17)$$

In both cases the parameters α_s are [3-6, 9]

$$\alpha_s = l + s \qquad (s = 0, 1, \ldots). \qquad (18)$$

The use of shape invariance to determine the energy eigenvalues and coordinate-space eigenfunctions of these two systems has been discussed in the literature [3-6, 9].

3. Shape invariance in the momentum representation

In this section we present the method of shape invariance in a form that is suitable for application to the momentum representation of spherically symmetric potentials in an angular momentum basis. The formulation is similar to that in the previous section but with some differences.

Instead of r and p_r we introduce operators $p=(\boldsymbol{p}\cdot\boldsymbol{p})^{1/2}$ and

$$r_p=\frac{1}{2}(\hat{\boldsymbol{p}}\cdot\boldsymbol{r}+\boldsymbol{r}\cdot\hat{\boldsymbol{p}})=\frac{1}{p}(\boldsymbol{p}\cdot\boldsymbol{r}+\mathrm{i}\hbar) \tag{19}$$

which satisfy the commutation relation (see [9, p 212] and [13])

$$[r_p, f(p)]=\mathrm{i}\hbar\frac{\mathrm{d}f}{\mathrm{d}p}. \tag{20}$$

In the momentum representation r_p can be expressed as the first-order differential operator

$$r_p=\frac{\mathrm{i}\hbar}{p}\frac{\mathrm{d}}{\mathrm{d}p}p. \tag{21}$$

Next we consider operators

$$B_\pm(p,\beta_0)=\pm\mathrm{i}r_pf(p)+\bar{W}(p,\beta_0) \tag{22}$$

where β_0 is a set of parameters and $\bar{W}$ and f are determined below (sections 4 and 5). (The ordering of r_p and f adopted in (22) is for later convenience.) We then construct operators which are second order in r_p, namely

$$\Lambda_\pm=B_\pm(\beta_0)B_\mp(\beta_0)=[r_pf(p)]^2+\bar{V}_\pm(p,\beta_0) \tag{23}$$

where

$$\bar{V}_\pm(p,\beta_0)=\bar{W}^2(p,\beta_0)\mp\hbar f(p)\frac{\mathrm{d}\bar{W}(p,\beta_0)}{\mathrm{d}p}. \tag{24}$$

The partner 'potentials' $\bar{V}_\pm$ are shape invariant if

$$\bar{V}_+(p,\beta_0)=\bar{V}_-(p,\beta_1)+\bar{R}(\beta_1) \tag{25}$$

where $\bar{R}$ is independent of p and $\beta_1=G(\beta_0)$.

Suppose Λ_- possesses a discrete spectrum and consider the eigenvalue equation

$$\Lambda_-\bar{\psi}_N(p,\beta_0)=\lambda_N\bar{\psi}_N(p,\beta_0)\qquad(N=0,1,\ldots) \tag{26}$$

where $\bar{\psi}_N$ is a normalized momentum-space eigenfunction. By analogy with (9) we suppose that

$$B_+(\beta_0)\bar{\psi}_0(p,\beta_0)=0. \tag{27}$$

Then $\lambda_0=0$.

If the shape invariance condition (25) is satisfied then the eigenvalues and momentum-space eigenfunctions of Λ_- are given by formulae which are similar to (10)–(12):

$$\lambda_N=\sum_{s=1}^{N}\bar{R}(\beta_s)\qquad(N=1,2,\ldots) \tag{10$'$}$$

$$B_-(\beta_0)\bar{\psi}_N(p,\beta_1)=\mathrm{e}^{\mathrm{i}\phi}\sqrt{\lambda_{N+1}}\,\bar{\psi}_{N+1}(p,\beta_0)\qquad(N=0,1,\ldots) \tag{11$'$}$$

$$\bar{\psi}_N(p,\beta_0)=\mathrm{e}^{-\mathrm{i}N\phi}\left[\prod_{s=1}^{N}(\lambda_N-\lambda_{s-1})\right]^{-1/2}B_-(\beta_0)B_-(\beta_1)\ldots B_-(\beta_{N-1})\bar{\psi}_0(p,\beta_N) \tag{12$'$}$$

for $N=1,2,\ldots$. Here $\beta_s=G(\beta_{s-1})$ and ϕ is a real constant.

The discussion so far has been formal, and we now give three examples to which the results (10′)-(12′) can be applied, namely the isotropic non-relativistic oscillator, a relativistic (Dirac) oscillator, and the Coulomb problem in an angular momentum basis.

4. Oscillators in an angular momentum basis

We start with a radial Hamiltonian for the isotropic non-relativistic oscillator expressed in a form which is quadratic in r_p, namely (see [13] and [9, p 213])

$$H_l=\frac{1}{2}m\omega^2 r_p^2+\bar{V}(p) \tag{28}$$

where

$$\bar{V}(p)=\frac{1}{2}m\omega^2\hbar^2 l(l+1)\frac{1}{p^2}+\frac{1}{2m}p^2. \tag{29}$$

Comparing (28) and (23), we take $f=\sqrt{\tfrac{1}{2}m}\,\omega$. 'Potentials' of the shape (29) can be generated from a 'superpotential'

$$\bar{W}(p,\beta_s)=\sqrt{\frac{1}{2}m}\;\hbar\omega(\beta_s+1)\frac{1}{p}-\frac{1}{\sqrt{2m}}p: \tag{30}$$

substituting (30) in (24) we see that if $\beta_0=l$ then

$$\bar{V}_-(p,\beta_0)=\bar{V}(p)-\hbar\omega(l+\tfrac{3}{2}). \tag{31}$$

Also, if

$$\beta_s=l+s \tag{32}$$

then $\bar{V}_\pm$ satisfy the shape invariance condition (25) with constant remainder

$$\bar{R}=2\hbar\omega. \tag{33}$$

In (23) let $\bar{V}_-$ be given by (31): comparing the result with (28) we see that

$$\Lambda_-=H_l-\hbar\omega(l+\tfrac{3}{2}). \tag{34}$$

From (10′) and (33) we have $\lambda_N=2N\hbar\omega$, and hence for the eigenvalues of H_l we obtain the familiar result $E_N=(2N+l+\tfrac{3}{2})\hbar\omega$, where $N=0,1,\ldots$ is the radial quantum number.

With $\bar{W}$ given by (30) and r_p by (21), the normalized solution to (27) is

$$\bar{\psi}_0(p,\beta_0)=\left[\frac{2^{\beta_0+2}}{p_0^3\pi^{1/2}(2\beta_0+1)!!}\right]^{1/2}\left(\frac{p}{p_0}\right)^{\beta_0}\exp\left(-\frac{p^2}{2p_0^2}\right) \tag{35}$$

where $p_0=\sqrt{m\hbar\omega}$ and $n!!=n(n-2)\ldots2$ (n even), $=n(n-2)\ldots1$ (n odd). The normalized momentum-space eigenfunctions for $N=1,2,\ldots$ can be obtained by substituting (35), with β_0 replaced by β_N, in (12′). The first few of these are found to be particular cases of

$$\bar{\psi}_N(p,\beta_0)=\mathrm{e}^{-\mathrm{i}N\phi}\left[\frac{2^{\beta_0+2-N}(2\beta_0+1+2N)!!}{p_0^3\pi^{1/2}N![(2\beta_0+1)!!]^2}\right]^{1/2}{}_1F_1(-N;\beta_0+\tfrac{3}{2};p^2/p_0^2)$$
$$\times\left(\frac{p}{p_0}\right)^{\beta_0}\exp\left(-\frac{p^2}{2p_0^2}\right) \tag{36}$$

where the confluent hypergeometric function ${}_1F_1$ is a polynomial of order N in p^2/p_0^2. The identification (36) can be extended to all non-negative integers N: we use (36) in (11′) to obtain

$$\left(1-\frac{u}{\beta_0+\frac{3}{2}}+\frac{u}{\beta_0+\frac{3}{2}}\frac{\mathrm{d}}{\mathrm{d}u}\right){}_1F_1(-N;\,\beta_0+\tfrac{5}{2};\,u)={}_1F_1(-N-1;\,\beta_0+\tfrac{3}{2};\,u)$$

($u=p^2/p_0^2$), which is a standard recurrence relation [14, p 507, entry 13.4.14].

We remark that shape invariance can also be applied to a relativistic oscillator known as the Dirac oscillator [15, 16]. This oscillator is described by a Dirac equation in which the interaction of a particle of rest mass m with an external potential is introduced by the (non-minimal) substitution $\boldsymbol{p}\to\boldsymbol{p}-\mathrm{i}m\omega\beta\boldsymbol{r}$, where ω is a constant and β is a Dirac matrix [15, 16]. Consequently the Dirac equation is linear in both $\boldsymbol{p}$ and $\boldsymbol{r}$. (There has been considerable interest in this oscillator, both for its algebraic properties [15–17] and for its possible application to QCD [18, 19].) If the spinor for the Dirac oscillator is written

$$\Psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\exp(-\mathrm{i}Et/\hbar)$$

then it can be shown that in an angular momentum basis the upper and lower elements of Ψ satisfy the eigenvalue equations [17]

$$\mathscr{H}\psi_\pm=\hbar\omega\left(\frac{E^2-m^2c^4}{2mc^2\hbar\omega}+k\pm\frac{1}{2}\right)\psi_\pm. \tag{37}$$

Here

$$\mathscr{H}=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2\boldsymbol{r}^2 \tag{38}$$

and $k=(-1)^{j+l-\frac{1}{2}}(j+\frac{1}{2})$ is the Dirac quantum number. Because $\mathscr{H}$ is the same as the Hamiltonian for an isotropic non-relativistic oscillator, it is clear that shape invariance can also be applied to the Dirac oscillator. The calculations of the energy eigenvalue E and the coordinate- and momentum-space eigenfunctions are similar to those discussed above for the non-relativistic oscillator and will not be repeated here: the results are the same as those obtained by other algebraic methods (see [17] where the coordinate- and momentum-space eigenfunctions and the phase relationships between the upper and lower elements of those eigenfunctions are given).

5. The Coulomb problem in an angular momentum basis

Our starting point is the radial Hylleraas equation for the Coulomb problem (see [13] and [9, p 215])

$$\left(\left[\frac{1}{\hbar}r_p(p^2-2ME)\right]^2+\frac{l(l+1)(p^2-2mE)^2}{p^2}\right)\bar{\psi}(p)=4p_0^2\bar{\psi}(p) \tag{39}$$

where $p_0=mk/\hbar$ and k is the constant in (15). We consider bound states ($E<0$) and define

$$u=p/\sqrt{-2mE}\qquad\qquad r_u=\sqrt{-2mE}\;r_p. \tag{40}$$

In terms of these (39) can be written

$$\bar{H}_l\bar{\psi}(u)=\varepsilon\bar{\psi}(u) \tag{41}$$

where

$$\bar{H}_l=\left[\frac{1}{\hbar}r_u(u^2+1)\right]^2+\bar{V}(u) \tag{42}$$

$$\bar{V}(u)=l(l+1)(u^2+u^{-2}) \tag{43}$$

$$\varepsilon=-\frac{2p_0^2}{mE}-2l(l+1). \tag{44}$$

The rest of the analysis is similar to that in the previous section for the oscillator. In applying the formulae in section 3 we replace p and r_p with u and r_u. Comparing (42) and (23) we take $f=(u^2+1)/\hbar$. With

$$\bar{W}(u,\beta_s)=-(\beta_s+1)(u-u^{-1}) \tag{45}$$

and

$$\beta_s=l+s \tag{46}$$

in (24) and (25) we find that

$$\bar{V}_-(u,\beta_0)=\bar{V}(u)-2(l+1)(l+2) \tag{47}$$

and that $\bar{V}_\pm$ satisfy the shape invariance condition (25) with remainder

$$\bar{R}(\beta_0)=8l+4=8\beta_0+4. \tag{48}$$

From (10′), (46) and (48) we obtain

$$\lambda_N=\sum_{s=1}^{N}(8\beta_s+4)=4N^2+8lN+8N. \tag{49}$$

In (23) let $\bar{V}_-$ be given by (47). Then

$$\Lambda_-=\bar{H}_l-2(l+1)(l+2). \tag{50}$$

From (44), (49) and (50) we have

$$4N^2+8lN+8N=-\frac{2p_0^2}{mE}-2l(l+1)-2(l+1)(l+2)$$

which yields the Bohr formula

$$E=-p_0^2/2m(N+l+1)^2 \tag{51}$$

where $N=0,1,\ldots$ is the radial quantum number.

With $\bar{W}$ given by (45) and r_p by (21), the normalized solution to (27) is

$$\bar{\psi}_0(p,\beta_0)=\left(\frac{2^{4\beta_0+6}(\beta_0+1)^4(\beta_0!)^2}{p_0^3 2\pi(2\beta_0+1)!}\right)^{1/2}\frac{u^{\beta_0}}{(u^2+1)^{\beta_0+2}}. \tag{52}$$

By substituting (52), with β_0 replaced by β_N, in (12′) we generate the normalized momentum-space eigenfunctions for $N=1,2,\ldots$. It is straightforward though tedious to show that the first few of these are particular cases of

$$\bar{\psi}_N(p,\beta_0)=e^{-iN\phi}\left(\frac{2^{4\beta_0+6}(\beta_0+1+N)^4(\beta_0!)^2(2\beta_0+1+N)!}{p_0^3 2\pi N![(2\beta_0+1)!]^2}\right)^{1/2}$$
$$\times{}_2F_1(-N,2\beta_0+2+N;\beta_0+\tfrac{3}{2};(u^2+1)^{-1})\frac{u^{\beta_0}}{(u^2+1)^{\beta_0+2}} \tag{53}$$

where u is given by (40) and (51), and the hypergeometric function ${}_2F_1$ is a polynomial of order N in $(u^2+1)^{-1}$. The validity of (53) for all non-negative integers N can be proved by using (53) in (11′). With $x=(u^2+1)^{-1}$ this yields

$$\left(1-2x+\frac{2(1-x)x}{2\beta_0+3}\frac{\mathrm{d}}{\mathrm{d}x}\right){}_2F_1(-N, 2\beta_0+4+N; \beta_0+\tfrac{5}{2}; x)$$
$$={}_2F_1(-N-1, 2\beta_0+3+N; \beta_0+\tfrac{3}{2}; x)$$

which is a standard recurrence relation (see [14, p 557, entry 15.2.9]).

In conclusion we remark that the class of problems to which shape invariance can be applied is, of course, larger for the coordinate representation than it is for the momentum representation. In particular, if one considers just s states then the centrifugal term in (13) is zero and there are additional potentials such as the Morse, Rosen–Morse, Pöschl–Teller, and Eckart potentials which can be treated by shape invariance in the coordinate representation [3–5]. We have not considered such problems here because even for s states it is not clear how to treat them by shape invariance in the momentum representation.

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